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QUASI-TRANSVERSE SHOCK WAVES IN AN ELASTIC MEDIUM IN THE CASE OF SPECIAL TYPES OF INITIAL STRAIN*

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Weak shock waves in isotropic elastic medium under an arbitrarily small initial strain were considered in /1,2/. The present paper deals with shock waves with higher order of symmetry, when there are special types of initial strain; all the results are obtained in explicit form.

1. Formulation of the problem. The investigations are carried out using the same formulation and the same degree of accuracy as in /1—3/. The general form of the elastic potential of the isotropic medium is given by the expression

$$\Phi = \rho_0 U(\varepsilon_{ij}, S) = {}^1/_2\,\lambda I_1^2 + \mu I_2 + \beta I_1 I_2 + \gamma I_3 + \delta I_1^3 + \xi I_2^2 + \rho_0 T_0 (S - S_0) + \text{const.}$$

$$I_1 = \varepsilon_{ii}, \quad I_2 = \varepsilon_{ij}\varepsilon_{ij}, \quad I_3 = \varepsilon_{ij}\varepsilon_{jk}\varepsilon_{ki}$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial w_i}{\partial \xi_j} + \frac{\partial w_j}{\partial \xi_i} + \frac{\partial w_k}{\partial \xi_i}\frac{\partial w_k}{\partial \xi_j}\right)$$

Here U is the internal energy, ρ_0 is the density in the stress-free state, T is the temperature, S is the entropy, ε_{ij} are the finite strain tensor components, w_i are the displacement tensor components, ξ_i are Lagrangian coordinates, and the coordinate system in the stress-free state is rectangular Cartesian. The axes of this system are chosen so that the

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planes $\xi_3 = \text{const}$ represent wave fronts at various instants, and $W = d\xi_3/dt$ is the wave velocity along the coordinate ξ_3. We prevent these planes from rotating by putting $\partial w_3/\partial\xi_1 = \partial w_3/\partial\xi_2 = 0$. The axes ξ_1 and ξ_2 are chosen so that $\partial w_1/\partial\xi_2 + \partial w_2/\partial\xi_1 = 0$ (the principal axes in the $\xi_1\xi_2$-plane). We eliminate the rotation of these axes by means of condition $\partial w_1/\partial\xi_2 - \partial w_2/\partial\xi_1 = 0$. Here ε_{12} vanishes in both the linear and non-linear approximation. At the shock wave front the only parameters undergoing a jump are $u = \partial w_1/\partial\xi_3$, $v = \partial w_2/\partial\xi_3$, $w = \partial w_3/\partial\xi_3$. The initial state of strain is described by the initial values of these quantities U, V, w_0 and by the strain components ε_{11}, ε_{22}, which remain unchanged at the discontinuity.

We know /1/ that in such media only one pair of quasilongitudinal and two pairs of quasi-transverse waves propagating on both sides of the axis ξ_3 may exist. Just as in /2/, we shall only consider quasi-transverse waves.

In the case of arbitrary initial strain, the equation of the shock adiabatics in the uv-plane has the following form for the quasi-transverse waves:

$$
\begin{aligned}
&(u^2 + v^2 - R^2)(Uv - Vu) + 2G(u - U)(v - V) = 0 &&(1.1)\\
&w = w_0 + b(R^2 - u^2 - v^2)/(\lambda + \mu)\\
&G = (2\mu + {}^3/_2\gamma)(\varepsilon_{22} - \varepsilon_{11})/\varkappa\\
&\varkappa = \mu + (\mu + \beta + {}^3/_2\gamma)^2/(\lambda + \mu) - 2\xi\\
&2b = \lambda + 2\mu + \beta + {}^3/_2\gamma,\quad R^2 = U^2 + V^2
\end{aligned}
$$

The condition of non-decrease in entropy and velocity of the shock wave are written in the form

$$8\rho_0 T_0(S - S_0) = -\varkappa\,[(u - U)^2 + (v - V)^2]\,(u^2 + v^2 - R^2) \geqslant 0 \tag{1.2}$$

$$
\begin{aligned}
&\rho_0 W^2 = \alpha_0 - {}^1/_2\varkappa\left\{u^2 + v^2 - R^2 + Uu + Vv + \frac{G(u-U)^2 - G(v-V)^2 + 2[U(u-U) + V(v-V)]^2}{(u-U)^2 + (v-V)^2}\right\} &&(1.3)\\
&\alpha_0 = \mu + 2bI_1^0 - (\mu + {}^3/_4\gamma)(\varepsilon_{11} + \varepsilon_{22})
\end{aligned}
$$

The condition of non-decrease in entropy (1.2) is satisfied in the uv-plane by the inside of the S-circle $u^2 + v^2 = R^2$ for media with $\varkappa > 0$, and the region outside this circle for media with $\varkappa < 0$. The conditions of evolution /1,2,4/ of the discontinuity

$$
\begin{aligned}
&\text{a)}\ c_2^- \leqslant W \leqslant c_2^+,\quad c_1^+ \leqslant W \leqslant c_3^- &&(1.4)\\
&\text{b)}\ c_1^- \leqslant W \leqslant c_1^+,\ 0 \leqslant W \leqslant c_2^-
\end{aligned}
$$

must hold in addition to (1.2) for the shock to exist.

Here c_i^-, c_i^+ are the characteristic velocities in the states ahead of and behind the shock respectively

$$
\begin{aligned}
&\rho_0(c_{1,2}^+)^2 = \alpha_0 - \varkappa\{u^2 + v^2 \pm {}^1/_2[(v^2 - u^2 - G)^2 + 4u^2v^2]^{1/2}\} &&(1.5)\\
&\rho_0(c_{1,2}^-)^2 = \alpha_0 - \varkappa\{R^2 \pm {}^1/_2[(V^2 - U^2 - G)^2 + 4U^2V^2]\}
\end{aligned}
$$

The $\pm$ signs are chosen so that $c_2 > c_1$. The two systems of inequalities (1.4) enable us to separate all shock waves into fast (a) and slow (b) waves.

Let us use the point $A(U, V)$ to represent in the uv-plane the initial state before the shock (Fig.1). The symmetry properties of the shock adiabatic /2/ clearly imply that it is sufficient to assume always that $G \geqslant 0$, $U \geqslant 0$, $V \geqslant 0$. In all figures thick lines depict the segments of the shock adiabatic satisfying simultaneously the conditions (1.2) and (1.4) for media with $\varkappa > 0$, and thick dashed lines are used for media with $\varkappa < 0$. Here we have $W = c_2^-$ at the points F, F', K, K', D; $W = c_1^-$ at the point L; $W = c_1^+$ at the points E, H; $W = c_2^+$ at the point J. In the present problem condition (1.4) was found to be stronger than (1.2), and the demand that the evolution conditions (1.4) be satisfied is sufficient for both sets of inequalities to hold.

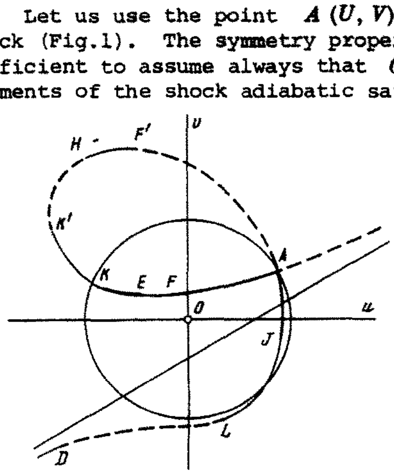

Fig.1

In the case of arbitrary initial strain the position of the ends of the evolution regions (points F, F', K, K', E, H, J, D, L) can only be shown qualitatively. On the other hand, if any of the initial strains $U, V, |\varepsilon_{11} - \varepsilon_{22}|$ are very small or altogether absent, the equation of the shock adiabatic degenerates into a circle intersected by a straight line and the position of the ends of the evolutionary segments can be found analytically. Below, we shall retain the notation used to indicate all end points in the general case. When there are no initial strains we obtain the problem which was studied in /5/.

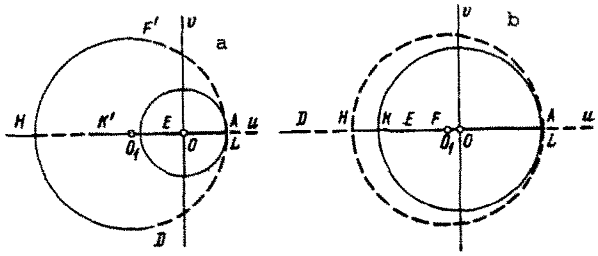

Fig.2

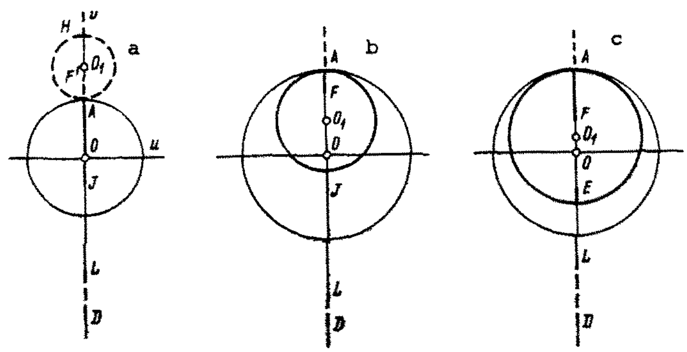

Fig.3

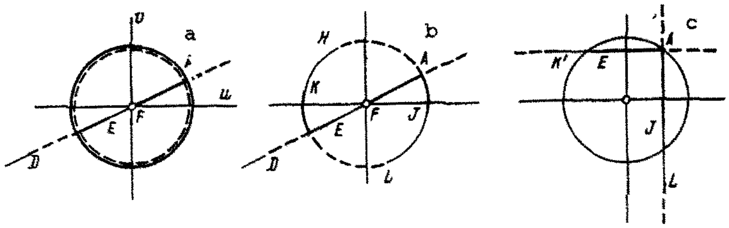

Fig.4

2. The case $V = 0$. We have $U > 0, G > 0$. The equation of the shock adiabatic decomposes into two equations

$$v = 0, \ (u + G/U)^2 + v^2 = (U + G/U)^2 \tag{2.1}$$

The above equations represent, in the uv-plane, the abscissa axis and a circle with centre at the point O_1: $u = -G/U < 0, v = 0$. Its radius $R_a = U + G/U$ is greater than the radius $R = U$ of the entropic circle (1.2). Both circles pass through the initial point $A\ (U, 0)$ and the entropic circle lies wholly within (2.1) (Fig.2).

For media with $\varkappa > 0$ the only physically feasible regions will be those lying on the segment of diameter $v = 0$ within the S-circle. The number and positions of these ranges depend on the initial strain. When $U^2 \leqslant 8G$, only a slow shock wave can exist and the corresponding segement of the adiabatic $U \geqslant u \geqslant -U/2$ (Fig.2a) is AE. When $U^2 > 8G$, a slow shock wave may exist on the segment AF of the abscissa $U \geqslant u \geqslant {}^1/_2\,[-U + (U^2 - 8G)^{1/2}]$ and a fast shock wave on the segment $KE: -{}^1/_2\,[U + (U^2 - 8G)^{1/2}] \leqslant u \leqslant -U/2$.

For media with $\varkappa < 0$ the conditions of evolution (1.4) are satisfied by the following parts of the adiabatic: the segment on the positive part of the abscissa $u \geqslant U$, $v = 0$ beginning at the point A (fast shock wave); if $U^2 < 2G$, then we also have a fast shock wave on the segment HK': $-(2G/U + U) \leqslant u \leqslant -2U$, $v = 0$; the slow wave on the circle (2.1) is represented by an arc which passes through the initial point A, is symmetric about the abscissa axis and its ends have the coordinate $u = -U^3/G$, i.e. they always lie in the second and fourth quadrants of the uv-plane (Fig.2a). In the limit as $U^2/G \to 0$, the arc ends are located on the v-axis. When $U^2 = 2G$, the ends of the arcs join and the whole circle (2.1) satisfies the conditions (1.2), (1.4). When $U^2 > 2G$, the slow wave has, apart from the complete circle, also the corresponding segment DH of the abscissas $-2U \leqslant u \leqslant -(2G/U + U)$ (Fig.2b).

3. The case $U = 0$. We have $V > 0, G > 0$. The equation of the shock adiabatic has the form

$$u = 0, \quad u^2 + (v - G/V)^2 = (V - G/V)^2 \tag{3.1}$$

This represents the whole of the ordinate axis and a circle of radius $R_a = |V - G/V|$ with centre at the point $O_1 (u = 0, v = G/V)$, passing through the initial point A $(0, V)$, at which it touches the entropic circle (1.2). When $G > V^2$, then for the media with $\varkappa > 0$ the segment of the ordinate extending from the initial point A to the point $v = -V/2$ fast wave) (Fig.3a) will be evolutionary. Three shock waves may exist for media with $\varkappa < 0$ under the same initial conditions: a slow wave at the segment of the oridnate axis where $V \leqslant v \leqslant {}^1/_2 [-V + (V^2 + 8G)^{1/2}]$, another slow wave on the segment DL of the same axis $-{}^1/_2 [V + (V^2 + 8G)^{1/2}] \leqslant v \leqslant -2V$; and a fast wave represented by the complete circle (3.1) and the positive part of the ordinate axis extending from the point H to the circle and beyond. Compared with the general case (Fig.1), here we have two arcs of the circle corresponding to the fast waves joined at the point A.

If $G < V^2$, then the whole circle (3.1) lies inside the entropic circle and A is their only common point. For materials with $\varkappa > 0$ condition (1.4) singles out the following segments of the adiabatic: the segment AF of the ordinate axis $V \geqslant v \geqslant {}^1/_2 [-V + (V^2 + 8G)^{1/2}]$ (slow wave), and the whole circle (3.1) and a segment of the adiabatic extending from the circle to the point $v = -V/2$ (fast wave). If $V^2 > 4G$, then the point $v = -V/2$ lies inside the circle (3.1) and serves as point E where $W(E) = c_1^+$ (Fig.3b). If on the other hand $4G > V^2 > G$, then the point $v = -V/2$ becomes the point J where $W(J) = c_2^+$ (Fig.3c). When $V^2 = 4G$, the adiabatic passes through the point $v = -V/2 = -\sqrt{G}$, and the characteristic velocities in it coincide $c_1^+ = c_2^+$. In studying simple waves /3/ the point $v = -\sqrt{G}$ in question was one of the singularities of the integral curves.

For media with $\varkappa < 0$ and the conditions $G < V^2$, the segments feasible for shock waves will be the segment on the positive part of the ordinate axis from the initial point onwards $V \leqslant v < \infty$ (fast wave) and the segment DL of the ordinate axis on which $-2V \leqslant v \leqslant -{}^1/_2 [V + (V^2 + 8G)^{1/2}]$ (slow wave). Clearly, when $G = V^2$, the points D and L merge and the segments DL and AF vanish, while the circle of the adiabatic (3.1) contracts to the point A. If $U = 0$ and $V = 0$ simultaneously, the adiabatic (1.1) degenerates into two straight lines coinciding with the u, v-coordinate axes, and the entropic circle (1.2) contracts to the point O. The state before the shock corresponds to the origin of coordinates. Clearly, under such prior strain no shock waves can exist in media with $\varkappa > 0$. In the case of media with $\varkappa < 0$, the state behind the shock wave corresponds to the whole abscissa axis (rapid wave) and the segment $-\sqrt{2G} \leqslant v \leqslant \sqrt{2G}$ of the ordinate axis (slow wave).

4. The case $G = 0$. Here $U > 0$, $V > 0$. The shock adiabatic (1.1) takes the form

$$u^2 + v^2 = R^2, \quad v = uV/U \tag{4.1}$$

The adiabatic again degenerates into a circle intersected by a straight line passing through the origin of coordinates in the direction of the former asymptotics (Fig.4a). The circle of the adiabatic coincides with the entropic circle (1.2); therefore we have $S = \text{const}$ everywhere on it. The circle yields a shock wave in which there is no increase in entropy for media with $\varkappa > 0$, as well as $\varkappa < 0$, and the velocity W coincides with one of the characteristic velocities (c_2^- for $\varkappa > 0$ and c_1^- for $\varkappa < 0$). The wave can also be regarded as a simple wave in the form of a step propagating at a constant rate without changing its form /3/. We see from (1.1) that the wave is purely transverse, and we have $w = w_0$.

The segments lying on the rectilinear part of the adiabatic, physically realizable in media with $\varkappa > 0$, will be the segments extending from the initial point A (U, V) to the origin of coordinates F (slow wave) and from the point E $(-U/2, -V/2)$ to the point $-U, -V$ fast wave). For media with $\varkappa < 0$ the segments will be represented by a ray originating at the initial point $v = uV/U$ (fast wave) and a segment of this straight line extending from the circle to the point D $(-2U, -2V)$ (slow wave) (Fig.4a).

Since G represents a product of initial deformation and elastic constants of the medium, it follows that the case $G = 0$ can be interpreted in various ways. It can imply either the absence of the deformation $\varepsilon_{11} - \varepsilon_{22} = 0$ in media with finite $\varkappa$ and $2\mu + {}^3/_2\gamma \neq 0$, or it can represent media of special type in which $\gamma = -{}^4/_3\mu$. In such media a part of the shock adiabatic will be a circle with centre at the origin, and $S = \text{const}$ for arbitrary initial strain.

5. The case $G \ll R^2$. Let us see whether the results obtained in Sect.4 for $G = 0$ can be arrived at by passing to the limit as $G \to 0$. We consider a value of G which is small, but not zero. We write $G/R^2 = \delta \ll 1$ and study the whole solution in terms of the expansion in δ, restricting ourselves to the first powers in δ. In this case the coordinates of all points

(F, K, E, H, J, D, L) can be determined exactly by analytical methods (when G is small, the points F' and K' are not present /2/). The shock adiabatic itself differs little from the corresponding curve for $G=0$. The coordinates of the points F, E and D differ little from the corresponding coordinates in Sect.4.

$$F\ (-2U^3\delta/R^2,\ 2V^3\delta/R^2)$$
$$E\ (-{}^1/_2\, U\,[1+12\,V^2\delta/R^2],\ -{}^1/_2 V\,[1-12U^2\delta/R^2])$$
$$D\ (-2U\,[1-3V^2\delta/R^2],\ -2V\,[1+3\,U^2\delta/R^2])$$

To find the coordinates of the remaining points we use the variable x which varies monotonically along the adiabatic /1/. We can assume that $x=\operatorname{tg}\varphi$ where φ is the polar angle in a system with centre at the initial point A, and the angle φ is measured from the first characteristic direction at the point A which for small δ is almost the same as the direction of the radius vector of A. Passing from x to Cartesian u, v-coordinates, we obtain for small δ the formulas

$$u = U - 2\,(U - Vx)/(1+x^2) + O\,(\delta),\quad v = V - 2\,(V + Ux)/(1+x^2) + O\,(\delta)$$

The x coordinates of the points K and L are found as the roots of the quadratic equation $W - c_i^- = 0$

$$x_{L,K} = \frac{U^2 - V^2 \pm (R^4 - U^2V^2)^{1/2}}{3UV} + O\,(\delta)$$

The point L always lies in the fourth quadrant below the point $U, -V$, and the point K lies in the second quadrant below the point $-U, V$. The position of the points depends on the initial state U, V. When $U\to 0$, both points K and L approach the abscissa axis, and when $V\to 0$, the ordinate axis. This agrees with the limiting cases $U=0$ or $V=0$ (Sect.2 and 3).

The coordinates of the points J and H are obtained as the two real roots of the equation

$$3x^4 - 4x^3\,(U^2 - V^2)/(UV) - 6x^2 - 1 + O\,(\delta) = 0$$

The roots were found by numerical methods. The point J lies in the fourth quadrant above the point $U, -V$, and point lies H in the second quadrant above the point $-U, V$. Both points J and H approach the abscissa axis as $V\to 0$, and the ordinate axis as $U\to 0$. Irrespective of the initial conditions, there are segments on the loop of the adiabatic on which the conditions of evolution (1.4) do not hold.

When $\delta\to 0$, the adiabatic becomes a circle intersected by its diameter. The points F, E, D pass to their limit positions corresponding to Sect.4 and the remaining evolutionary segments EK, AJ, AH occupy only a part of the adiabatic circle (Fig.4b) and not the whole circle as in Sect.4.

6. The case $G \gg R^2$.

When R^2/G are small, the size of the loop in the shock adiabatic becomes of the order of G/R. Limiting ourselves as before to the region of order $\sqrt{G}$, we find that since $G/R \gg \sqrt{G}$, the shock adiabatic in this region degenerates into two intersecting straight lines $u=U, v=V$. The evolution conditions on these lines are satisfied by the following segments: for materials with $\varkappa>0$ we have the segment $AE: U\geqslant u \geqslant -U/2$ on the straight line $v=V$ (slow wave) and the segment $AJ: V\geqslant v\geqslant -V/2$ on the line $u=U$ (fast wave); for materials with $\varkappa<0$ we have the segments $-\infty<u\leqslant -2U$ and $U\leqslant u<\infty$ on the line $v=V$ (fast waves) and the segments $V\leqslant v<\infty$ and $-\infty<v\leqslant -2V$ on the line $u=U$ (slow waves) (Fig.4c).

Just as in Sect.4, the restrictions imposed on G hold either because of the initial strain, or because of other properties (constants) of the medium itself. We can assume that $|\varepsilon_{11} - \varepsilon_{22}| \gg R^2$ when $(2\mu + {}^3/_2\gamma)/\varkappa$, is finite, or that $\varkappa\to 0$.

7. Media with $\varkappa = 0$.

Should we find a material in which $\varkappa=0$, we shall find (see (1.1)–(1.3)) that when $(2\mu + {}^3/_2\gamma)\,(\varepsilon_{22} - \varepsilon_{11})$ are finite, shock waves will propagate with constant velocity W without an increase in entropy S. The shock adiabatic will be represented by two straight lines $u=U, v=V$ whose points will all satisfy conditions (1.2) and (1.4).

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